

Large N field theory and AdS tachyons

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

JHEP04(2009)020

(<http://iopscience.iop.org/1126-6708/2009/04/020>)

[The Table of Contents](#) and [more related content](#) is available

Download details:

IP Address: 80.92.225.132

The article was downloaded on 03/04/2010 at 10:35

Please note that [terms and conditions](#) apply.

Large N field theory and AdS tachyons

Elli Pomoni and Leonardo Rastelli

*C.N. Yang Institute for Theoretical Physics, Stony Brook University,
Stony Brook, NY 11794-3840, U.S.A.*

E-mail: elli.pomoni@stonybrook.edu , leonardo.rastelli@stonybrook.edu

ABSTRACT: In non-supersymmetric orbifolds of $\mathcal{N} = 4$ super Yang-Mills, conformal invariance is broken by the logarithmic running of double-trace operators — a leading effect at large N . A tachyonic instability in AdS_5 has been proposed as the bulk dual of double-trace running. In this paper we make this correspondence more precise. By standard field theory methods, we show that the double-trace beta function is quadratic in the coupling, to all orders in planar perturbation theory. Tuning the double-trace coupling to its (complex) fixed point, we find conformal dimensions of the form $2 \pm i b(\lambda)$, as formally expected for operators dual to bulk scalars that violate the stability bound. We also show that conformal invariance is broken in perturbation theory if and only if dynamical symmetry breaking occurs. Our analysis is applicable to a general large N field theory with vanishing single-trace beta functions.

KEYWORDS: Tachyon Condensation, AdS-CFT Correspondence, $1/N$ Expansion

ARXIV EPRINT: [0805.2261](https://arxiv.org/abs/0805.2261)

Contents

1	Introduction	1
2	Renormalization of double-trace couplings	4
2.1	Double-trace renormalization to all orders	5
2.1.1	The $\lambda = 0$ case	6
2.1.2	The general case	8
3	Double-trace running and dynamical symmetry breaking	11
3.1	Running coupling	12
3.2	Effective potential	13
3.3	Stability versus conformal invariance	15
4	AdS/CFT	17
4.1	Two examples	18
4.2	Classical flat directions and instability	19
5	Discussion	19

1 Introduction

Conformal invariant quantum field theories in four dimensions are interesting both theoretically and for potential phenomenological applications. While perturbatively finite supersymmetric QFTs have been known for a long time [1] and a vast zoo of non-perturbative supersymmetric examples was discovered during the duality revolution of the 1990s, only few non-supersymmetric, interacting CFTs in $d = 4$ are presently known.¹

The AdS/CFT correspondence [3–5] seems to offer an easy route to several more examples. A well-known construction [6, 7] starts by placing a stack of N D3 branes at an orbifold singularity \mathbb{R}^6/Γ . In the decoupling limit one obtains the duality between an orbifold of $\mathcal{N} = 4$ SYM by $\Gamma \subset \text{SU}(4)_R$ and Type IIB on $AdS_5 \times S^5/\Gamma$. Supersymmetry is completely broken if $\Gamma \not\subset \text{SU}(3)$, but since the AdS factor of the geometry is unaffected by the orbifold procedure, conformal invariance appears to be preserved, at least for large N . However, in the absence of supersymmetry one may worry about possible instabilities [8].

On the string theory side of the duality, one must draw a distinction [9] according to whether the orbifold action has fixed points or acts freely on S^5 . If Γ has fixed points, there are always closed string tachyons in the twisted sector. If Γ acts freely, the twisted strings are stretched by a distance of the order of the S^5 radius R ; the would-be tachyons

¹Large N Bank-Zaks [2] fixed points come to mind.

are then massive for large enough R (strong 't Hooft coupling λ), but it is difficult to say anything definite about small R .

On the field theory side, a perturbative analysis at small λ reveals that conformal invariance is *always* broken, regardless of whether the orbifold is freely acting or not [10, 11]. The inheritance arguments of [8, 12] guarantee that the orbifold theory is conformal in its single-trace sector: at large N , all couplings of marginal single-trace operators have vanishing beta functions. However, even at leading order in N , there are non-zero beta functions for double-trace couplings of the form

$$\delta S = f \int d^4x \mathcal{O}\bar{\mathcal{O}}, \tag{1.1}$$

where \mathcal{O} is a twisted single-trace operator of classical dimension two [9–11, 13, 14]. Conformal invariance could still be restored, if all double-trace couplings f_k had conformal fixed points. It turns out that this is never the case in the one-loop approximation [10, 11]. So for sufficiently small λ , all non-supersymmetric orbifolds of $\mathcal{N} = 4$ break conformal invariance.

It is natural to associate this breaking of conformal invariance with the presence of tachyons in the dual AdS theory [10]. By an AdS tachyon, we mean a scalar field that *violates* the Breitenlohner-Freedman bound [15]:

$$\text{For a tachyon, } m^2 < m_{\text{BF}}^2 = -\frac{4}{R^2}. \tag{1.2}$$

One is then led to speculate [10] that even for freely acting orbifolds, some of the twisted states must become tachyonic for λ smaller than some critical value λ_C . The conjectural behavior of $m^2(\lambda)$ for a “tachyon” in a freely acting orbifold theory is shown in figure 1. A related viewpoint [9] links the tachyonic instability in the bulk theory with a perturbative Coleman-Weinberg instability in the boundary theory. From this latter viewpoint however, it seems at first that whether Γ is freely acting or not makes a difference even at weak coupling [9]: if Γ has fixed points, the quantum-generated double-trace potential destabilizes the theory along a classical flat direction; if Γ is freely acting, the symmetric vacuum appears to be stable, because twisted operators have zero vevs along classical flat directions.

In this paper we make the correspondence between double-trace running and bulk tachyons more precise. Taken at face value, an AdS_5 tachyon would appear to be dual to a boundary operator with *complex* conformal dimension of the form

$$\Delta = 2 \pm i b, \quad b = \sqrt{|m^2 R^2 + 4|}. \tag{1.3}$$

We are going to find a formal sense in which this is correct, and a prescription to compute the tachyon mass $m^2(\lambda)$ from the boundary theory. In principle this prescription could be implemented order by order in λ and allow to test the conjectural picture of figure 1. We also show that the perturbative CW instability is present if and only if conformal invariance is broken, independently of the tree-level potential, and thus independently of whether the orbifold is freely acting or not.

Our analysis applies to the rather general class of large N theories “conformal in their single-trace sector”. We consider non-supersymmetric, classically conformal field theories

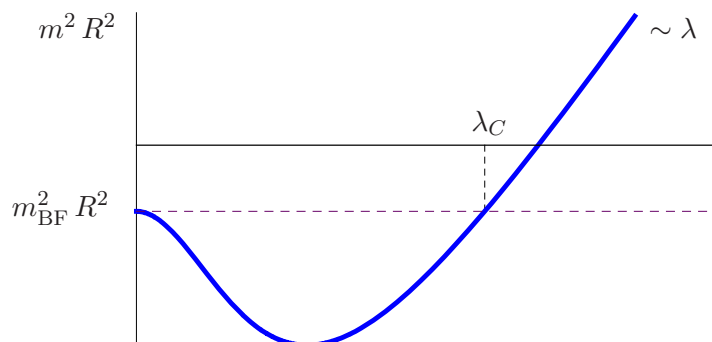


Figure 1. Proposal for the qualitative behavior of a “tachyon” mass in a freely acting orbifold, as a function of the ’t Hooft coupling λ . The field is an actual tachyon (violating the BF stability bound) for $\lambda < \lambda_C$. See section 4.1 for more comments.

with lagrangian of the standard single-trace form $\mathcal{L} = N \text{Tr} [\dots]$. Denoting collectively by λ the single-trace couplings that are kept fixed in the large N limit,² we assume that $\beta_\lambda \equiv \mu \frac{\partial}{\partial \mu} \lambda = 0$ at large N . Generically however, perturbative renormalizability forces the addition of double-trace couplings of the form (1.1), where $\mathcal{O} \sim \text{Tr} \phi^2$ is a single trace operator of classical dimension two. Thus it is essential to compute the double-trace beta functions β_f to determine whether or not conformal invariance is maintained in the quantum theory. Our main technical results are expressions for β_f , for the conformal dimension $\Delta_{\mathcal{O}}$ and for the effective potential $\mathcal{V}(\varphi)$, valid to all orders in planar perturbation theory.

Besides orbifolds of $\mathcal{N} = 4$ SYM, other examples of large N theories conformal in their single-trace sector are certain non-supersymmetric continuous deformations of $\mathcal{N} = 4$ SYM [16–18]. One can also contemplate theories with adjoint *and* fundamental matter, where the instability arises in the mesonic sector and is dual to an *open* string tachyon. A detailed analysis of such an “open string” example will appear in a forthcoming paper [19]. Somewhat surprisingly, conformal invariance turns out to be broken in all concrete cases of non-supersymmetric “single-trace conformal” theories that have been studied so far. There is no a priori reason of why this should be the case in general. A more systematic search for conformal examples is certainly warranted.

We should also mention from the outset that independently of the perturbative instabilities which are the focus of this paper, non-supersymmetric orbifold theories may exhibit a non-perturbative instability akin to the decay of the Kaluza-Klein vacuum [20] (see also [21]). For a class of freely acting \mathbb{Z}_{2k+1} orbifolds, at large coupling λ the decay-rate per unit volume scales as [20]

$$\Gamma_{\text{decay}} \sim k^9 e^{-N^2/k^8} \Lambda^4, \tag{1.4}$$

where Λ is a UV cut-off. This instability is logically distinct and parametrically different from the tree-level tachyonic instability. It is conceivable that a given orbifold theory may be stable in a window of couplings $\lambda_C < \lambda < \lambda_{\text{KK}}$ intermediate between a critical value

² In the example of an orbifold of $\mathcal{N} = 4$ SYM, $\lambda = g_{\text{YM}}^2 N$ is the usual ’t Hooft coupling.

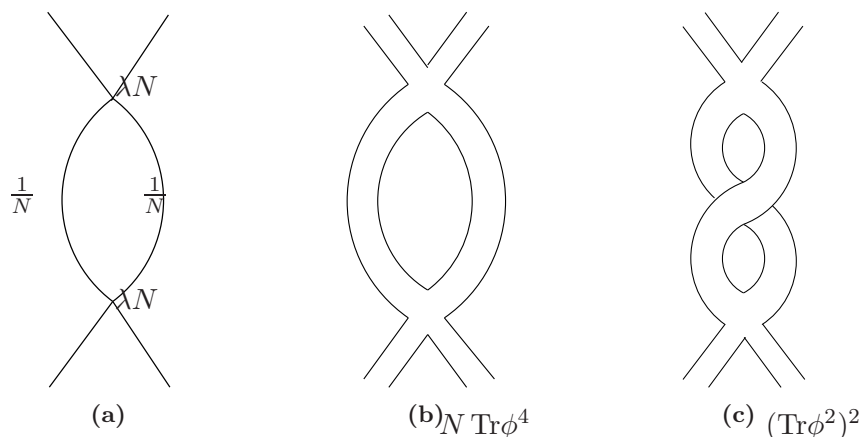


Figure 2. One-loop contributions to the effective action from a diagram with two quartic vertices. Each vertex contributes a factor of λN and each propagator a factor of $1/N$, as indicated in (a). There are two ways to contract color indices: a single-trace structure (b), or a double-trace structure (c).

λ_C where the “tachyon” is lifted (figure 1) and another critical value λ_{KK} where the non-perturbative instability sets in.

Multitrace deformations in the context of the AdS/CFT correspondence have been investigated in several papers, beginning with [22–26].

The paper is organized as follows. In section 2 we study the renormalization of a general field theory conformal in the single-trace sector and derive expressions for β_f and $\Delta_{\mathcal{O}}$ valid to all orders in planar perturbation theory. In section 3 we study the behavior of the running coupling $f(\mu)$ and the issue of stability of the quantum effective potential $\mathcal{V}(\varphi)$. In section 4 we make our proposal for the computation of the tachyon mass $m^2(\lambda)$ from the dual field theory. We illustrate the prescription in a couple of examples and make some remarks on flat directions in freely acting orbifold theories. We conclude in section 5 discussing a few open problems.

2 Renormalization of double-trace couplings

We are interested in large N , non-supersymmetric field theories in four dimensions. We start with a conformally invariant classical action of the standard single-trace form. Schematically,

$$S_{\text{ST}}[N, \lambda] = \int d^4x N \text{Tr} [(D\phi)^2 + \psi D\psi + (DA)^2 + \lambda \phi^4 + \dots], \quad (2.1)$$

where ϕ, ψ, A are $N \times N$ matrix-valued scalar, spinor and gauge fields. We have written out the sample interaction term $N\lambda \text{Tr} \phi^4$ to establish our notation for the couplings: we denote collectively by λ the couplings in S_{ST} that are kept fixed in the large N limit.

Generically, the action (2.1) is not renormalizable as it stands, because extra double-trace interactions are induced by quantum corrections. It is an elementary but underappreciated fact that double-trace renormalization is a *leading* effect at large N . For

example, consider the contribution to the effective action from one-loop diagrams with two quartic scalar vertices (figure 2). Schematically,

$$\int d^4x N \lambda \text{Tr} \phi^4(x) \int d^4y N \lambda \text{Tr} \phi^4(y) \sim \lambda^2 \log \Lambda \int d^4z [N \text{Tr} \phi^4 + (\text{Tr} \phi^2)^2] . \quad (2.2)$$

The single-trace term $N \text{Tr} \phi^4$ renormalizes a coupling already present in the action (2.1). The double-trace term $(\text{Tr} \phi^2)^2$ forces the addition of an extra piece to the bare action,

$$S = S_{\text{ST}} + S_{\text{DT}}, \quad S_{\text{DT}} = \int d^4x f_0 (\text{Tr} \phi^2)^2, \quad f_0 \sim \lambda^2 \log \Lambda . \quad (2.3)$$

It is crucial to realize that S_{ST} and S_{DT} are of the same order in the large N limit, namely $O(N^2)$. For S_{ST} , one factor of N is explicit and the other arises from the trace; for S_{DT} , each trace contributes one factor of N .

In the following, we specialize to theories for which the single-trace couplings do not run in the large N limit, $\beta_\lambda = \mu \frac{\partial}{\partial \mu} \lambda = O(1/N)$. In particular the single-trace contribution in (2.2) is canceled when we add all the relevant Feynman diagrams. This is what happens in orbifolds of $\mathcal{N} = 4$ SYM. Twisted single-trace couplings cannot be generated in the effective action, since they are charged under the quantum symmetry, while untwisted single-trace couplings are not renormalized, since they behave as in the parent theory by large N inheritance. However, neither argument applies to double-trace couplings of the form $f \mathcal{O}_g \mathcal{O}_g^\dagger$, where $\mathcal{O}_g = \text{Tr}(g\phi^2)$ is a twisted single-trace operator of classical dimension two.³ Such double-trace couplings will be generated in perturbation theory.

In the rest of this section, we analyze the general structure of double-trace renormalization.

2.1 Double-trace renormalization to all orders

The beta function for the double-trace coupling (1.1) was computed at one loop in [10],

$$\beta_f \equiv \mu \frac{\partial}{\partial \mu} f = v^{(1)} f^2 + 2\gamma^{(1)} \lambda f + a^{(1)} \lambda^2 . \quad (2.4)$$

This result applies to any theory conformal in its single-trace sector. Here $v^{(1)}$ is the normalization of the single-trace operator $\mathcal{O} \sim \text{Tr} \phi^2$, defined as

$$\langle \mathcal{O}(x) \bar{\mathcal{O}}(y) \rangle = \frac{v^{(1)}}{2\pi^2(x-y)^4} . \quad (2.5)$$

The quantity $\gamma^{(1)} \lambda$ is the one-loop contribution to the anomalous dimension of \mathcal{O} from the single-trace interactions. The double-trace interaction also contributes to the renormalization of \mathcal{O} , so that the full result for its one-loop anomalous dimension is

$$\gamma_{\mathcal{O}} = \gamma^{(1)} \lambda + v^{(1)} f . \quad (2.6)$$

Some representative Feynman diagrams contributing to β_f are shown in figure 3. Our goal is to generalize these results to all orders in planar perturbation theory.

³ Here $\text{Tr} = \text{Tr}_{\text{SU}(\Gamma|N)}$ and $g \in \Gamma$.

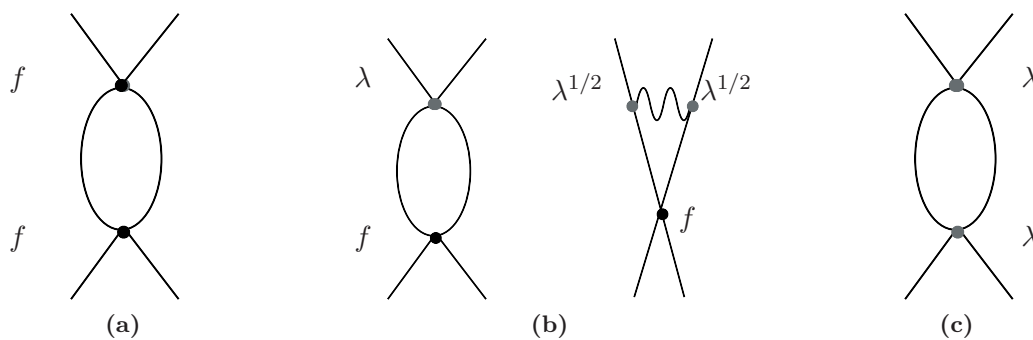


Figure 3. Sample diagrams contributing to β_f at one loop: (a) $v^{(1)}f^2$; (b) $2\gamma^{(1)}\lambda f$; (c) $a^{(1)}\lambda^2$.

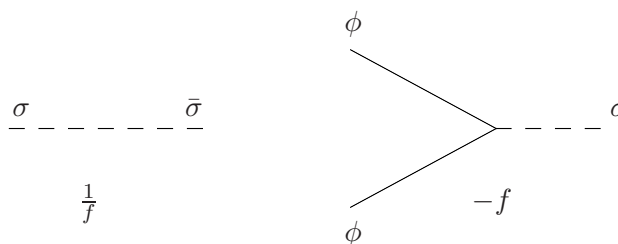


Figure 4. Feynman rules for (2.8).

2.1.1 The $\lambda = 0$ case

Let us first practice with the simple situation where the single-trace part of the action is free.⁴ The total lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{ST}}^{\text{free}} + \mathcal{L}_{\text{DT}}, \quad \mathcal{L}_{\text{DT}} = f \mathcal{O} \bar{\mathcal{O}}. \tag{2.7}$$

The discussion of the large N theory is facilitated by a Hubbard-Stratonovich transformation. We introduce the auxiliary complex scalar field σ and write the equivalent form for the double-trace interaction,⁵

$$\mathcal{L}_{\text{DT}} = -f\sigma\bar{\sigma} + f\sigma\bar{\mathcal{O}} + f\bar{\sigma}\mathcal{O}. \tag{2.8}$$

The obvious Feynman rules are displayed in figure 4. The renormalization program is carried out as usual, by adding to the tree-level lagrangian (2.8) local counterterms, which we parametrize as

$$\delta\mathcal{L}_{\text{DT}} = -(Z_2 - 1)f\sigma\bar{\sigma} + (Z_3 - 1)(f\sigma\bar{\mathcal{O}} + f\bar{\sigma}\mathcal{O}). \tag{2.9}$$

The one-particle irreducible structures that may contain divergences are $\Gamma_{\sigma\bar{\sigma}}$, $\Gamma_{\sigma\phi\phi}$ and $\Gamma_{\phi\phi\phi\phi}$. The quartic vertex $\Gamma_{\phi\phi\phi\phi}$ is in fact subleading in the large N limit, as illustrated in figure 5 in a one-loop example. The leading contributions to the scalar four-point function

⁴ The calculation of β_f for this case already appears in [24].

⁵For ease of notation we suppress possible flavor indices for \mathcal{O} and σ .

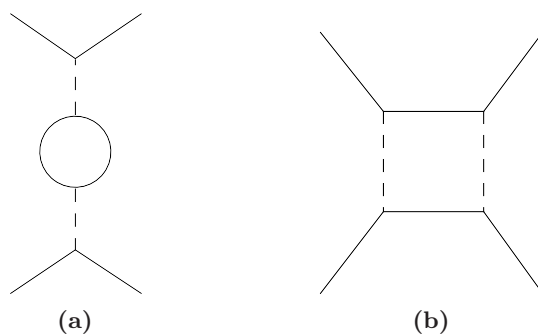


Figure 5. Diagram (a) is leading at large N , of order $O(1)$, but it is reducible. Diagram (b) is irreducible but it is subleading at large N , of order $O(1/N^2)$.

contain cuttable σ propagators. This is an example of a general fact that we will use repeatedly: 1PI diagrams with internal σ propagators are subleading for large N . Indeed, adding internal σ lines increases the number of ϕ propagators, which are suppressed by $1/N$.

The upshot is that while for finite N (2.8) is not renormalizable as written (we need to add an explicit $\mathcal{O}\bar{\mathcal{O}}$ counterterm), for large N it is.

From the Feynman rules, we immediately find

$$\Gamma_{\sigma\bar{\sigma}}(x, y) = f Z_2 \delta(x - y) + Z_3^2 f^2 \langle \mathcal{O}(x)\bar{\mathcal{O}}(y) \rangle_{f=0}, \tag{2.10}$$

$$\Gamma_{\sigma\phi\phi}(x; y, z) = -f Z_3 \langle \mathcal{O}(x)\phi(y)\phi(z) \rangle_{f=0}^{1PI}. \tag{2.11}$$

Since we are assuming for now that the single-trace action is free, the $f = 0$ correlators appearing above are given by their tree-level expressions. The three-point function $\langle \mathcal{O}\phi\phi \rangle_{f=0}^{1PI}$ is simply a constant,

$$\Gamma_{\sigma\phi\phi} = -f Z_3 \cdot \text{const}. \tag{2.12}$$

Clearly no renormalization of the $\sigma\phi\phi$ vertex is needed and we can set $Z_3 = 1$. On the other hand, the two-point function

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(0) \rangle_{f=0} \equiv \frac{v}{2\pi^2 x^4} \tag{2.13}$$

requires renormalization, since its short-distance behavior is too singular to admit a Fourier transform. We adopt the elegant scheme of differential renormalization [27, 28]. The singularity is regulated by smearing the scalar propagator,

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(0) \rangle_{f=0} = \frac{v}{2\pi^2} \frac{1}{(x^2 + \epsilon^2)^2}, \tag{2.14}$$

where ϵ is a short distance cutoff. Introducing a dimensionful constant μ , one may separate out the divergence as follows,

$$\frac{v}{2\pi^2} \frac{1}{(x^2 + \epsilon^2)^2} \xrightarrow{\epsilon \rightarrow 0} -\frac{v}{8\pi^2} \square \frac{\ln x^2 \mu^2}{x^2} - v \ln \mu \epsilon \delta(x). \tag{2.15}$$

The first term is the renormalized two-point function: it is finite (Fourier transformable) if one interprets the Laplacian as acting to the left under the integral sign. The constant μ plays the role of the renormalization scale. Back in (2.10), we take the Z -factors to be

$$Z_2 = 1 + vf \log \mu \epsilon, \quad Z_3 = 1, \quad (2.16)$$

and find the renormalized correlator

$$\Gamma_{\sigma\bar{\sigma}}(x, y) = f\delta(x - y) - \frac{vf^2}{8\pi^2} \square \frac{\ln \mu^2(x - y)^2}{(x - y)^2}. \quad (2.17)$$

We are now in the position to calculate β_f and the anomalous dimension $\gamma_{\mathcal{O}}$ of the single-trace operator.⁶ The renormalized two-point function satisfies the Callan-Symanzik equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_f \frac{\partial}{\partial f} - 2\gamma_{\mathcal{O}} \right] \Gamma_{\sigma\bar{\sigma}} = 0. \quad (2.18)$$

Recalling the identity

$$\mu \frac{\partial}{\partial \mu} \left[-\frac{1}{8\pi^2} \square \frac{\ln \mu^2 x^2}{x^2} \right] = \delta(x), \quad (2.19)$$

we see that the CS equation implies

$$2f\beta_f - 2\gamma_{\mathcal{O}}f^2 = 0 \quad (2.20)$$

$$\beta_f - 2\gamma_{\mathcal{O}}f + vf^2 = 0, \quad (2.21)$$

the first condition arising for $x \neq y$ and the second from the delta function term. Incidentally, the CS equation for $\Gamma_{\sigma\phi\phi}$, namely

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_f \frac{\partial}{\partial f} - \gamma_{\mathcal{O}} - 2\gamma_{\phi} \right] \Gamma_{\sigma\phi\phi} = 0, \quad \gamma_{\phi} = 0, \quad (2.22)$$

immediately gives $\beta_f = f\gamma_{\mathcal{O}}$, equivalent to (2.20). Solving the linear system, we find

$$\beta_f = vf^2, \quad \gamma_{\mathcal{O}} = vf. \quad (2.23)$$

These are exact results (all orders in f) in the large N theory. The essential point, borne out by the auxiliary field trick, is that for $\lambda = 0$ the only primitively divergent diagram is the one-loop renormalization of the σ propagator.

2.1.2 The general case

As we take $\lambda \neq 0$, we face the complication that the version of the theory with the auxiliary field, equation (2.8), is not renormalizable as it stands, since an explicit quartic term $\mathcal{O}\bar{\mathcal{O}}$ is regenerated by the interactions. We are led to consider the two-parameter theory

$$\mathcal{L}^{(2)}(g, h) \equiv \mathcal{L}_{\text{ST}} - g\sigma\bar{\sigma} + g\sigma\bar{\mathcal{O}} + g\bar{\sigma}\mathcal{O} + h\mathcal{O}\bar{\mathcal{O}}. \quad (2.24)$$

⁶ Note that $\gamma_{\mathcal{O}}$ coincides with γ_{σ} , since connected correlation functions of σ are equal (for separated points) to connected correlation functions of \mathcal{O} .

Comparing with the original form of the lagrangian without auxiliary field,

$$\mathcal{L}^{(1)}(f) \equiv \mathcal{L}_{\text{ST}} + f\mathcal{O}\bar{\mathcal{O}}, \quad (2.25)$$

we have the equivalence

$$\mathcal{L}^{(1)}(g+h) \sim \mathcal{L}^{(2)}(g,h). \quad (2.26)$$

(We leave implicit the dependence of $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ on the single-trace couplings λ and on N .) Clearly,

$$\beta_f(g+h) = \beta_g(g,h) + \beta_h(g,h), \quad (2.27)$$

where β_f is the beta function for the coupling f in theory (2.25), and β_g and β_h are the beta functions for the couplings g and h in theory (2.24). It may appear that not much is gained by considering the more complicated lagrangian $\mathcal{L}^{(2)}(g,h)$, but in fact the auxiliary field trick still provides a useful reorganization of large N diagrammatics. Our strategy is to work in the theory defined by $\mathcal{L}^{(2)}(g,h)$, but *in the limit that the renormalized quartic coupling $h \rightarrow 0$* .

We need not discuss explicitly the renormalization of the single-trace part of the action. For large N , the 1PI diagrams that renormalize the couplings in $\mathcal{L}_{\text{ST}}(\lambda)$ are independent of g , because leading diagrams at large N do not contain internal σ lines. Since we are also taking $h \rightarrow 0$, this implies that the renormalization of $\mathcal{L}_{\text{ST}}(\lambda)$ proceeds independently of $\mathcal{L}_{\text{DT}}^{(2)}$. We recall that by assumption, $\mathcal{L}_{\text{ST}}(\lambda)$ is such that $\beta_\lambda = 0$ for large N .

To discuss the renormalization of $\mathcal{L}_{\text{DT}}^{(2)}(g, h \rightarrow 0)$, we parametrize the counterterms as

$$\delta\mathcal{L}_{\text{DT}}^{(2)} = -(Z_2 - 1)g\sigma\bar{\sigma} + (Z_3 - 1)(g\sigma\bar{\mathcal{O}} + g\bar{\sigma}\mathcal{O}) + (Z_4 - 1)h\mathcal{O}\bar{\mathcal{O}}. \quad (2.28)$$

As we have emphasized, even for $h \rightarrow 0$ a quartic counterterm $(Z_4 - 1)h\mathcal{O}\bar{\mathcal{O}}$ is needed in order to cancel the divergence of $\Gamma_{\phi\phi\phi\phi}$. We can use again the fact that for large N , $\Gamma_{\phi\phi\phi\phi}$ is independent of g (recall figure 5). Hence for $h \rightarrow 0$ the quartic counterterm can only depend on the single-trace coupling λ ,

$$\lim_{h \rightarrow 0} (Z_4 - 1)h = f(\lambda, \epsilon, \mu). \quad (2.29)$$

It follows that the corresponding beta function is only a function of λ ,

$$\beta_h(g, h = 0) = a(\lambda). \quad (2.30)$$

In orbifolds of $\mathcal{N} = 4$ SYM, λ is the usual 't Hooft coupling, and $a(\lambda)$ has a perturbative expansion of the form

$$a(\lambda) = \sum_{L=1}^{\infty} a^{(L)}\lambda^{L+1}, \quad (2.31)$$

where L is the number of loops.

The analysis of the two remaining primitively divergent structures, $\Gamma_{\sigma\bar{\sigma}}$ and $\Gamma_{\sigma\phi\phi}$, proceeds similarly as in the $\lambda = 0$ case, with a few extra elements. We have (for $h = 0$),

$$\Gamma_{\sigma\bar{\sigma}}(x, y) = gZ_2 \delta(x - y) + Z_3^2 g^2 \langle \mathcal{O}(x)\bar{\mathcal{O}}(y) \rangle_{g=h=0}, \quad (2.32)$$

$$\Gamma_{\sigma\phi\phi}(x; y, z) = -gZ_3 \langle \mathcal{O}(x)\phi(y)\phi(z) \rangle_{g=h=0}^{1PI}. \quad (2.33)$$

From the last equation, we see that the factor Z_3 has the role of renormalizing the composite operator \mathcal{O} in the theory with $g = h = 0$,

$$\mathcal{O}_{g=h=0}^{\text{ren}} \equiv Z_3(\lambda, \mu, \epsilon) \mathcal{O}. \quad (2.34)$$

The dependence of $\mathcal{O}_{g=h=0}^{\text{ren}}$ on the renormalization scale μ is given by

$$\mu \frac{\partial}{\partial \mu} \mathcal{O}_{g=h=0}^{\text{ren}} = -\gamma(\lambda) \mathcal{O}_{g=h=0}^{\text{ren}}, \quad (2.35)$$

where $\gamma(\lambda)$ is, by definition, the anomalous dimension of the single-trace operator in the theory where we set to zero the double-trace couplings. The two-point function of $\mathcal{O}_{g=h=0}^{\text{ren}}$ takes then the standard form

$$\langle \mathcal{O}^{\text{ren}}(x)\mathcal{O}^{\text{ren}}(0) \rangle_{g=h=0} = \frac{v(\lambda)}{2\pi^2} \frac{\mu^{-2\gamma(\lambda)}}{x^{4+2\gamma(\lambda)}}, \quad x \neq 0. \quad (2.36)$$

We have indicated that the normalization v will in general depend on λ . In orbifolds of $\mathcal{N} = 4$, $v(\lambda)$ and $\gamma(\lambda)$ have perturbative expansions of the form

$$v(\lambda) = \sum_{L=1}^{\infty} v^{(L)} \lambda^{L-1}, \quad \gamma(\lambda) = \sum_{L=1}^{\infty} \gamma^{(L)} \lambda^L. \quad (2.37)$$

The expression (2.36) is not well-defined at short distance and needs further renormalization, which we perform again in the differential renormalization scheme. We first expand

$$\frac{\mu^{-2\gamma}}{x^{4+2\gamma}} = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!} \frac{\log^n \mu^2 x^2}{x^4}, \quad (2.38)$$

and then renormalize each term of the series using the substitutions [29]

$$\frac{\log^n \mu^2 x^2}{x^4} = -\frac{n!}{4} \square \sum_{k=1}^{n+1} \frac{1}{k!} \frac{\log^k \mu^2 x^2}{x^2}. \quad (2.39)$$

These are exact identities for $x \neq 0$ and provide the required modification of the behavior at $x = 0$, if one stipulates that free integration by parts is allowed under the integral sign.

Back in (2.32), we have⁷

$$\Gamma_{\sigma\bar{\sigma}}(x, 0) = gZ_2 \delta(x) + g^2 \langle \mathcal{O}^{\text{ren}}(x)\mathcal{O}^{\text{ren}}(0) \rangle_{g=h=0} \quad (2.40)$$

$$= g \delta(x) - g^2 \frac{v}{8\pi^2} \sum_{n=0}^{\infty} (-\gamma)^n \square \sum_{k=1}^{n+1} \frac{1}{k!} \frac{\log^k(\mu^2 x^2)}{x^2}. \quad (2.41)$$

⁷The value of Z_2 is defined implicitly by this equation. As in the $\lambda = 0$ case, we could introduce a short-distance cutoff ϵ and then choose $Z_2(\epsilon, \mu)$ such that the final result (2.41) for the fully renormalized correlator is obtained.

The CS equation,

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} - 2\gamma_{\mathcal{O}} \right] \Gamma_{\sigma\bar{\sigma}} = 0, \quad (2.42)$$

gives as before two conditions, one for $x \neq 0$ and one from the delta function term. For $x \neq 0$, we may simply use the naive expression (2.36) for the correlator, and we find

$$2g\beta_g - 2\gamma_{\mathcal{O}}g^2 - 2\gamma g^2 = 0. \quad (2.43)$$

It is easy to check that the same condition follows from the CS for $\Gamma_{\sigma\phi\phi}$. On the other hand, terms proportional to $\delta(x)$ in (2.42) arise either from the explicit $g\delta(x)$ in $\Gamma_{\sigma\bar{\sigma}}$, or when the μ derivative hits the $k = 1$ terms of the series,

$$0 = \beta_g - 2\gamma_{\mathcal{O}}g + g^2v \sum_{n=0}^{\infty} (-\gamma)^n = \beta_g - 2\gamma_{\mathcal{O}}g + \frac{g^2v}{1+\gamma}. \quad (2.44)$$

The solution of the linear system (2.43), (2.44) is

$$\gamma_{\mathcal{O}} = \gamma + \frac{vg}{1+\gamma}, \quad \beta_g = \frac{vg^2}{1+\gamma} + 2g\gamma. \quad (2.45)$$

We can finally evaluate β_f in the original theory (2.25). From

$$\beta_f(f) = \beta_g(g=f, h=0) + \beta_h(f, h=0), \quad (2.46)$$

we find

$$\beta_f = \frac{v(\lambda)}{1+\gamma(\lambda)} f^2 + 2\gamma(\lambda) f + a(\lambda). \quad (2.47)$$

This is the sought generalization of the one-loop result (2.4) originally found in [10]. The expression for the full conformal dimension of the single-trace operator is

$$\Delta_{\mathcal{O}} = 2 + \gamma_{\mathcal{O}}(f, \lambda) = 2 + \gamma(\lambda) + \frac{v(\lambda)}{1+\gamma(\lambda)} f. \quad (2.48)$$

The boxed equations are valid to all orders in large N perturbation theory.

3 Double-trace running and dynamical symmetry breaking

The beta function of the double-trace coupling remains quadratic in f , to all orders in planar perturbation theory. This simplification allows to draw some general conclusions about the behavior of the running coupling and the stability of the Coleman-Weinberg potential. While the essential physics is already visible in the one-loop approximation, it seems worthwhile to pursue a general analysis.

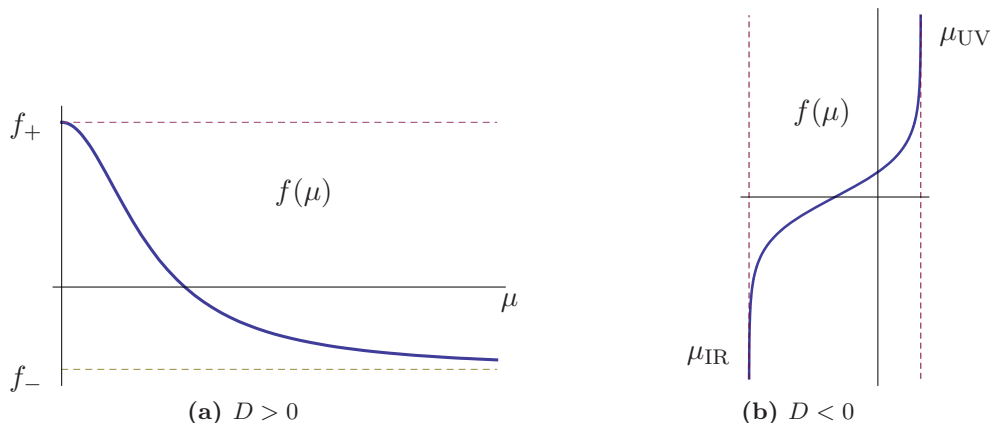


Figure 6. The two qualitative behaviors of the running coupling $f(\mu)$ for $D > 0$ and $D < 0$.

3.1 Running coupling

We need to distinguish two cases, according to whether the quadratic equation

$$\beta_f = \frac{v(\lambda)}{1 + \gamma(\lambda)} f^2 + 2\gamma(\lambda) f + a(\lambda) = 0 \tag{3.1}$$

has real or complex zeros. We define the discriminant $D(\lambda)$,

$$D(\lambda) \equiv \gamma(\lambda)^2 - \frac{a(\lambda)v(\lambda)}{1 + \gamma(\lambda)}, \tag{3.2}$$

and the square root of $|D|$,

$$b(\lambda) \equiv \sqrt{|D(\lambda)|}. \tag{3.3}$$

From (2.31), (2.37), $b(\lambda)$ has a perturbative expansion of the form

$$b(\lambda) = b^{(1)}\lambda + b^{(2)}\lambda^2 + \dots \tag{3.4}$$

- *Positive discriminant*

If $D > 0$, (3.1) has real solutions

$$f_{\pm} = -\frac{\gamma}{\tilde{v}} \pm \frac{b}{\tilde{v}}, \quad \tilde{v} \equiv \frac{v}{1 + \gamma}. \tag{3.5}$$

In this case we can maintain conformal invariance in the quantum theory by tuning f to one of the two fixed points. Since $v > 0$ (the two-point function of \mathcal{O} is positive by unitarity), we see that f_- is UV stable and f_+ IR stable. The differential equation for the running coupling,

$$\mu \frac{\partial}{\partial \mu} f(\mu) = \beta_f(f(\mu)), \tag{3.6}$$

is easily solved to give

$$f(\mu) = \frac{\left(\frac{\mu}{\mu_0}\right)^{2b} f_- + f_+}{\left(\frac{\mu}{\mu_0}\right)^{2b} + 1}. \quad (3.7)$$

The function $f(\mu)$ is plotted on the left in figure 6. The running coupling interpolates smoothly between the IR and the UV fixed points.

- *Negative discriminant*

If $D < 0$ there are no fixed points for real f and conformal invariance is broken in the quantum theory. The solution of (3.6) is

$$f(\mu) = -\frac{\gamma}{\tilde{v}} + \frac{b}{\tilde{v}} \tan \left[\frac{b}{\tilde{v}} \ln(\mu/\mu_0) \right], \quad \tilde{v} \equiv \frac{v}{1+\gamma}. \quad (3.8)$$

There are Landau poles both in the UV and in the IR, at energies

$$\mu_{\text{IR}} = \mu_0 \exp \left(-\frac{\pi \tilde{v}}{2b} \right) \cong \mu_0 \exp \left(-\frac{\pi v^{(1)}}{2b^{(1)} \lambda} \right) \quad (3.9)$$

$$\mu_{\text{UV}} = \mu_0 \exp \left(\frac{\pi \tilde{v}}{2b} \right) \cong \mu_0 \exp \left(\frac{\pi v^{(1)}}{2b^{(1)} \lambda} \right). \quad (3.10)$$

The behavior of $f(\mu)$ is plotted on the right in figure 6.

3.2 Effective potential

The running of the double-trace coupling f and the generation of a quantum effective potential for the scalar fields are closely related. We wish to make this relation precise.

Let us consider a spacetime independent vev for the scalars,

$$\langle \phi_a^{i b} \rangle = \varphi T_a^{i b}. \quad (3.11)$$

We have picked some direction in field space specified by the tensor $T_a^{i b}$, where i is a flavor index and $a, b = 1, \dots, N$ are color indices. We need not assume that it is a classical flat direction. With no loss of generality we take $\varphi \geq 0$.

We now go through the textbook renormalization group analysis of the quantum effective potential $\mathcal{V}(\varphi)$. The RG equation reads

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_f \frac{\partial}{\partial f} - \gamma_\phi \varphi \frac{\partial}{\partial \varphi} \right] \mathcal{V}(\varphi, \mu, f, \lambda) = 0, \quad (3.12)$$

where $\gamma_\phi(\lambda)$ is the anomalous dimension of the scalar field ϕ . Note that for large N , $\gamma_\phi(\lambda)$ is independent of f . Writing (3.12) as

$$\mathcal{V}(\varphi, \mu, f, \lambda) \equiv \varphi^4 U(\varphi/\mu, f, \lambda), \quad \left[\varphi \frac{\partial}{\partial \varphi} - \frac{\beta_f}{1+\gamma_\phi} \frac{\partial}{\partial f} + \frac{4\gamma_\phi}{1+\gamma_\phi} \right] U = 0, \quad (3.13)$$

one finds that the most general solution takes the form

$$\mathcal{V}(\varphi, \mu, f, \lambda) = \varphi^4 \left(\frac{\varphi}{\mu} \right)^{-\frac{4\gamma_\phi}{1+\gamma_\phi}} U_0(\hat{f}(\varphi), \lambda), \quad (3.14)$$

where $\hat{f}(\mu)$ satisfies

$$\mu \frac{\partial}{\partial \mu} \hat{f}(\mu) = \frac{\beta_f(\hat{f})}{1 + \gamma_\phi}. \quad (3.15)$$

In general, the arbitrary function $U_0(\hat{f}, \lambda)$ is found order by order by comparing with explicit perturbative results. In our case, because of large N , the double-trace coupling contributes to the effective potential only at tree-level. This is again a consequence of the fact that 1PI diagrams with internal σ lines are suppressed. Moreover, by assumption the single-trace quartic term $N\lambda\text{Tr}\phi^4$ is not renormalized at large N , so that the *explicit* λ dependence of $U_0(\hat{f}, \lambda)$ is also exhausted by the tree-level contribution. There is of course an implicit λ dependence in \hat{f} , as clear from (3.15), (2.47). The full tree-level contribution to the effective potential is

$$\mathcal{V}_{\text{tree}}(\varphi) = N\lambda\text{Tr}\phi^4 + f \mathcal{O}\bar{\mathcal{O}} = N^2(C_{\text{ST}}\lambda + C_{\text{DT}}f) \varphi^4, \quad (3.16)$$

where C_{ST} and C_{DT} are some non-negative proportionality constants of order one.⁸ If the vev is taken along a classical flat direction of the single-trace lagrangian, then $C_{\text{ST}} = 0$, but we need not assume this is the case. Thus

$$U_0(\hat{f}, \lambda) = N^2(C_{\text{ST}}\lambda + C_{\text{DT}}\hat{f}). \quad (3.17)$$

The final result for the large N effective potential is

$$\mathcal{V}(\varphi) = N^2 \mu^{\frac{4\gamma_\phi}{1+\gamma_\phi}} \left[C_{\text{ST}} \lambda + C_{\text{DT}} \hat{f}(\varphi) \right] \varphi^{\frac{4}{1+\gamma_\phi}}. \quad (3.18)$$

Ordinarily, at a fixed order in perturbation theory the RG improved effective potential can be trusted in the range of φ such that the running coupling $\hat{f}(\varphi)$ is small. In our case, $\mathcal{V}(\varphi)$ receives no higher corrections in \hat{f} , so it appears that (3.18), being the full non-perturbative answer, may have a broader validity.

Let us make contact with the explicit one-loop expression of the effective potential. To this order,

$$\tilde{v}(\lambda) \cong v^{(1)}, \quad \gamma(\lambda) \cong \gamma^{(1)}\lambda, \quad a(\lambda) \cong a^{(1)}\lambda^2, \quad \gamma_\phi \cong \gamma_\phi^{(1)}\lambda, \quad (3.19)$$

and the expansion of (3.18) gives

$$\begin{aligned} \mathcal{V}_{1\text{-loop}}(\varphi) = & \quad (3.20) \\ N^2 \varphi^4 \log \left(\frac{\varphi}{\mu} \right) \cdot & \left[v^{(1)} f^2 C_{\text{DT}} + 2f\lambda(\gamma^{(1)} - 2\gamma_\phi^{(1)})C_{\text{DT}} + \lambda^2(a^{(1)}C_{\text{DT}} - 4\gamma_\phi^{(1)}C_{\text{ST}}) \right]. \end{aligned}$$

Each term has an obvious diagrammatic interpretation.

⁸We are suppressing flavor indices: $N\lambda\text{Tr}\phi^4$ in (3.16) is a shortcut for the scalar potential of the single-trace lagrangian \mathcal{L}_{ST} , which we require to be bounded from below. Then $C_{\text{ST}} \geq 0$. On the other hand, positivity of C_{DT} is clear from (3.16), since $\mathcal{O}\bar{\mathcal{O}}$ is a positive quantity.

3.3 Stability versus conformal invariance

Armed with the general form (3.18) of the large N effective potential, we can investigate the stability of the symmetric vacuum at $\varphi = 0$. Since the single-trace coupling λ does not run, we can treat it as an external parameter. For given λ , the functions $a(\lambda)$, $\tilde{v}(\lambda)$, $\gamma(\lambda)$ and $\gamma_\phi(\lambda)$ are just constant parameters that enter the expression for $\mathcal{V}(\varphi)$.

The qualitative behavior of $\mathcal{V}(\varphi)$ is dictated by the discriminant $D(\lambda)$. Comparing (3.15) with (3.6), we see that $\hat{f}(\varphi)$ behaves just as $f(\varphi)$, up to some trivial rescaling of coefficients by $1/(1 + \gamma_\phi)$. We consider again the two cases:

- *Positive discriminant*

For $D > 0$, the running coupling is given by

$$\hat{f}(\varphi) = \frac{\left(\frac{\varphi}{\mu}\right)^{2\hat{b}} f_- + f_+}{\left(\frac{\varphi}{\mu}\right)^{2\hat{b}} + 1}, \quad \hat{b} \equiv \frac{b}{1 + \gamma_\phi}. \quad (3.21)$$

The constant solutions $\hat{f}(\varphi) = f_\pm$ are obtained as degenerate cases for $\mu \rightarrow 0$ and $\mu \rightarrow \infty$. In the generic case, the effective potential is bounded by the two functions (we set $\mu \equiv 1$)

$$N^2 (C_{\text{ST}}\lambda + C_{\text{DT}}f_+) \varphi^{\frac{4}{1+\gamma_\phi}} \leq \mathcal{V}(\varphi) \leq N^2 (C_{\text{ST}}\lambda + C_{\text{DT}}f_-) \varphi^{\frac{4}{1+\gamma_\phi}}, \quad (3.22)$$

where the lower bound is attained for $\varphi \rightarrow 0$ and the upper bound for $\varphi \rightarrow \infty$. Recall from (3.5) that $f_- < f_+$, with f_- always negative. If

$$C_{\text{ST}}\lambda + C_{\text{DT}}f_+ > 0, \quad (3.23)$$

then $\varphi = 0$ is at least a local minimum, otherwise it is a global maximum and the potential is unbounded from below. Condition (3.23) is simply the requirement that the tree-level potential (3.16) be bounded from below when f is set to its IR fixed point f_+ . If (3.23) holds, it is also permissible to simply pick the constant solution $\hat{f}(\varphi) = f_+$. Then \mathcal{V} is monotonically increasing and $\varphi = 0$ is the global minimum. In the generic case (3.21), we need the stronger condition

$$C_{\text{ST}}\lambda + C_{\text{DT}}f_- > 0 \quad (3.24)$$

to ensure that the potential is bounded from below. Then $\varphi = 0$ is the global minimum.

In view of the comments below (3.18), we believe that this analysis has general validity. It is certainly valid for $\lambda \ll 1$, since then $f_\pm \sim \lambda + O(\lambda^2)$, and the effective coupling $\hat{f}(\varphi) \ll 1$ for every value of φ .

In summary, barring pathological cases where the potential is unbounded from below, for $D > 0$ the vacuum $\varphi = 0$ is stable and dynamical symmetry breaking does not occur.

- *Negative discriminant*

If $D < 0$, the effective potential reads, in units $\mu \equiv 1$,

$$\mathcal{V}(\varphi) = N^2 \left[C_{\text{ST}}\lambda + C_{\text{DT}}\hat{f}(\varphi) \right] \varphi^{\frac{4}{1+\gamma_\phi}}, \quad \hat{f}(\varphi) = -\frac{\gamma}{\tilde{v}} + \frac{b}{\tilde{v}} \tan\left(\frac{b}{\tilde{v}} \log \varphi\right). \quad (3.25)$$

The theory only makes sense as an effective field theory for energy scales intermediate between the two Landau poles, $\mu_{\text{IR}} = e^{-\frac{\pi}{2b}} \ll \varphi \ll \mu_{\text{UV}} = e^{+\frac{\pi}{2b}}$. The potential ranges between minus infinity at μ_{IR} and plus infinity at μ_{UV} . A little algebra shows that $\mathcal{V}(\varphi)$ is either a monotonically increasing function, or it admits a local maximum and a local minimum. Local extrema exist if

$$\lambda \frac{C_{\text{ST}}}{C_{\text{DT}}} - \frac{\gamma}{\tilde{v}} < \frac{1}{1+\gamma_\phi} - \frac{b^2(1+\gamma_\phi)}{4\tilde{v}^2}, \quad (3.26)$$

with the potential always negative at the local minimum,

$$\mathcal{V}(\varphi_{\text{min}}) < 0. \quad (3.27)$$

From (3.19), (3.3), we see that (3.26) is always obeyed for sufficiently small λ . The value of the running coupling at the minimum can be expanded for $\lambda \ll 1$,

$$\hat{f}(\varphi_{\text{min}}) = -\alpha \lambda + \left(-\frac{a^{(1)}}{4} + \gamma_\phi^{(1)} \alpha - \frac{v^{(1)}}{4} \alpha^2 \right) \lambda^2 + O(\lambda^3), \quad \alpha \equiv \frac{C_{\text{ST}}}{C_{\text{DT}}}. \quad (3.28)$$

For small λ , $\hat{f}(\varphi_{\text{min}})$ is also small, the local minimum can be trusted, and dynamical symmetry breaking occurs. If the vev is taken along a flat direction for the single-trace potential, namely if $C_{\text{ST}} = 0$, then the double-trace coupling at the new vacuum is of order $O(\lambda^2)$, which is perhaps the more familiar behavior — as in the original analysis of massless scalar electrodynamics [30]. From (3.26), (3.27), (3.28), we find that for small λ symmetry breaking occurs even if the tree level single-trace potential does not vanish ($C_{\text{ST}} \neq 0$).

We take the liberty to belabor this conclusion, giving an alternative derivation. One can first expand the effective potential to lowest non-trivial order,

$$\mathcal{V}(\varphi) \cong N^2 [C_{\text{ST}}\lambda + C_{\text{DT}}\hat{f}(\mu)] + \mathcal{V}_{1\text{-loop}}(\varphi), \quad (3.29)$$

with $\mathcal{V}_{1\text{-loop}}$ given by (3.20). In looking for the minimum, $\mathcal{V}'(\varphi_{\text{min}}) = 0$, $\mathcal{V}''(\varphi_{\text{min}}) > 0$, it is convenient to set the renormalization scale $\mu \equiv \varphi_{\text{min}}$. Then we just solve for $\hat{f}(\varphi_{\text{min}})$ and easily reproduce (3.28). This is a consistent procedure provided we can find a renormalization trajectory where $\hat{f}(\varphi_{\text{min}})$ takes the value (3.28). A glance at figure 6 shows that yes, we can set \hat{f} to any prescribed value. Finally, since (3.28) happens to be small for λ small, the whole analysis can be trusted in perturbation theory.

The inequality (3.26) can be satisfied also if λ is of order one, in which case $\hat{f}(\varphi)$ is of order one. In view of our remarks about the non-perturbative validity of $\mathcal{V}(\varphi)$, it seems plausible that the local minimum can also be trusted in this case.

4 AdS/CFT

We have used standard field theory arguments to characterize the two possible behaviors for a large N theory conformal in its single-trace sector. Either all double-trace beta functions admit real zeros, and then the symmetric vacuum is stable and conformal invariance is preserved; or at least one beta function has no real solutions, and then conformal invariance is broken and dynamical symmetry breaking occurs.

We now give a reinterpretation of these results in light of the AdS/CFT correspondence. Even for negative discriminant, we insist in solving for the zeros of the double-trace beta function,

$$f_{\pm} = -\frac{\gamma}{\tilde{v}} \pm \frac{\sqrt{D}}{\tilde{v}}. \tag{4.1}$$

Setting $f = f_{\pm}$, the full conformal dimension (2.48) of the single-trace operator \mathcal{O} reads

$$\Delta_{\mathcal{O}} = 2 + \gamma + \tilde{v}f_{\pm} = 2 + \gamma - \gamma \pm \sqrt{D} = 2 \pm \sqrt{D}. \tag{4.2}$$

So at the fixed point, the anomalous dimension of \mathcal{O} is either real if $D > 0$ or *purely imaginary* if $D < 0$. This is just as expected from the AdS/CFT formula

$$\Delta_{\mathcal{O}} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2} = 2 \pm \sqrt{4 + m^2 R^2}, \tag{4.3}$$

where m is the mass of the AdS_5 scalar field dual to \mathcal{O} , if we identify

$$m^2(\lambda)R^2 = m_{\text{BF}}^2 R^2 + D(\lambda) = -4 + D(\lambda). \tag{4.4}$$

For $D > 0$, we are in the standard situation of real coupling constant, real anomalous dimension and dual scalar mass above the stability bound, $m^2 > m_{\text{BF}}^2$. We propose to take (4.4) at face value even when $D < 0$. If $m^2 < m_{\text{BF}}^2$, the AdS bulk vacuum is unstable. Similarly, if $D < 0$, the field theory conformal-invariant vacuum is unstable. Equation (4.4) gives the precise relation between the two instabilities. The proper treatment of both the bulk and the boundary theory would be to expand around the stable minimum. But in stating that the AdS scalar has a certain mass $m^2 < m_{\text{BF}}^2$, we are implicitly quantizing the bulk theory in an AdS invariant way. The dual statement is to formally quantize the boundary theory in a conformal invariant way, around the symmetric minimum $\varphi = 0$, by tuning the coupling to the complex fixed point $f = f_+$ (or f_-). At either fixed point, the operator dimension is complex,

$$\Delta_{\mathcal{O}} = 2 \pm i b. \tag{4.5}$$

The discriminant $D(\lambda) = \gamma(\lambda)^2 - a(\lambda)\tilde{v}(\lambda)$ is a purely field-theoretic quantity. In principle (4.4) is a prescription to compute the tachyon mass from the field theory, at least order by order in perturbation theory. It would be interesting to see if integrability techniques [31] are applicable to this problem, though the fact that \mathcal{O} is a “short” operator may represent a challenge. For now we may compare field theory results at weak coupling with the strong coupling behavior predicted by the gravity side. Let us look at a couple of examples.

4.1 Two examples

Expanding (4.4) to one-loop order,

$$m^2(\lambda)R^2 = -4 + D(\lambda) = -4 + \left[(\gamma^{(1)})^2 - a^{(1)}v^{(1)} \right] \lambda^2 + O(\lambda^3). \quad (4.6)$$

The coefficients $v^{(1)}$, $\gamma^{(1)}$, and $a^{(1)}$ were computed in [10, 11] for several orbifolds of $\mathcal{N} = 4$ SYM. Obtaining the corresponding m^2 is an exercise in arithmetic.

As a first illustration, take the \mathbb{Z}_2 orbifold theory that arises on a stack of N electric and N magnetic D3 branes of Type 0B string theory. There are twisted scalars in the $\mathbf{20}'$ and $\mathbf{1}$ representations of $SU(4)_R$. From the results in [10, 13], one finds

$$m_{\mathbf{20}'}^2 R^2 \cong -4 - \frac{\lambda^2}{8\pi^4} + O(\lambda^3), \quad m_{\mathbf{1}}^2 R^2 \cong -4 - \frac{23\lambda^2}{64\pi^4} + O(\lambda^3). \quad (4.7)$$

Since this orbifold has fixed points on the S^5 (it fixes the whole sphere), we expect these masses to remain negative below the stability bound for all λ , with the asymptotic behavior

$$m^2(\lambda)R^2 \sim -\frac{R^2}{\alpha'} = -\lambda^{1/2}, \quad \lambda \rightarrow \infty. \quad (4.8)$$

Let us also consider a simple class of non-supersymmetric freely acting orbifold, \mathbb{Z}_k orbifold with $SU(3)$ global symmetry [10]. The \mathbb{Z}_k action is

$$z_i \rightarrow \omega_k^n z_i, \quad \omega_k \equiv e^{\frac{2\pi i}{k}}, \quad n = 1, \dots, k, \quad (4.9)$$

where z_i , $i = 1, 2, 3$ are the three complex coordinates of $\mathbb{R}^6 = \mathbb{C}^3$. The orbifold is freely acting for k odd, and breaks supersymmetry for $k > 3$. Let us focus on the \mathbb{Z}_5 case. There are twisted operators $\mathcal{O}_{\mathbf{8},n}$, $\mathcal{O}_{\mathbf{1},n}$, with $n = 1, 2$, in the octet and singlet of the $SU(3)$ flavor group. It turns out that in the one-loop approximation the $n = 1$ operators have positive discriminant, while the $n = 2$ operators have negative discriminant. From the results of [10], one calculates

$$m_{\mathbf{8},2}^2 R^2 \cong -4 - \frac{\sqrt{5}-1}{640\pi^4} \lambda^2 + O(\lambda^3), \quad m_{\mathbf{1},2}^2 R^2 \cong -4 - \frac{7\sqrt{5}-1}{1600\pi^4} \lambda^2 + O(\lambda^3). \quad (4.10)$$

The conjectural behavior of $m^2(\lambda)$ for freely acting orbifolds is plotted in figure 1 in the introduction. The one-loop calculation (4.10) gives the second derivative at $\lambda = 0$. For large λ , these states correspond to highly stretched strings on the S^5 . The asymptotic behavior should thus be

$$m^2(\lambda)R^2 \sim \frac{R^4}{\alpha'^2} \sim \lambda, \quad \lambda \rightarrow \infty. \quad (4.11)$$

Figure 1 plots the simplest interpolation between the small and large λ limits. It would be very interesting to compute the $O(\lambda^3)$ corrections to (4.10): this picture suggests that they should be positive.

4.2 Classical flat directions and instability

The \mathbb{Z}_{2k+1} freely-acting orbifolds serve as an illustration of another point – classical flat directions are immaterial in our context. The classical moduli space of the theory is $(\mathbb{C}^3/\mathbb{Z}_{2k+1})^N/S_N$. In the brane picture this corresponds to the positions of the N D3 branes on the orbifold space $\mathbb{C}^3/\mathbb{Z}_{2k+1}$. The flat directions are parametrized by vevs for the bifundamental scalars (there are no adjoints). Along the flat directions, all twisted operators have zero vev.

As emphasized in [9], this is the case in general for freely acting orbifolds: they have no adjoint scalars and hence no classical branch along which the twisted operators could develop a vev. However, this does not imply that the symmetric vacuum is stable. On the contrary, we have seen in section 3.3 that dynamical symmetry breaking occurs at small coupling whenever $D < 0$, irrespective of the classical potential. Since one can always find a double-trace coupling with $D < 0$, whether the orbifold is freely acting or not [11], we conclude that freely acting orbifolds also have a CW instability which drives into condensation a twisted operator, $\langle \mathcal{O} \rangle \neq 0$. The instability occurs away from the flat directions.

This reconciles the proposal of [10], which relates bulk tachyons with the breaking of conformal invariance, with the general viewpoint of [9], which relates them to the Coleman-Weinberg instability. A detailed analysis of the CW instability in some examples of freely acting orbifolds has been pursued by [32].

5 Discussion

The logarithmic running of double-trace couplings $f\mathcal{O}\bar{\mathcal{O}}$, where $\mathcal{O} \sim \text{Tr}\phi^2$, is a general feature of large N field theories that contain scalar fields. In this paper we have studied the renormalization of double-trace couplings in theories that have vanishing single-trace beta functions at large N . We have derived general expressions for the double-trace beta function β_f , the conformal dimension $\Delta_{\mathcal{O}}$ and the effective potential $\mathcal{V}(\varphi)$. The main point is that β_f is a quadratic function of f (and $\Delta_{\mathcal{O}}$ a linear function of f), to all-orders in planar perturbation theory, with coefficients that depend on the single-trace couplings λ .

Double-trace running plays an important role in non-supersymmetric examples of the AdS/CFT correspondence. We have related the discriminant $D(\lambda)$ of β_f to the mass $m^2(\lambda)$ of the bulk scalar dual to the single-trace operator \mathcal{O} . If $D(\lambda) < 0$, the bulk scalar is a tachyon; on the field theory side, conformal invariance is broken and dynamical symmetry breaking occurs.

The authors of [11] considered orbifolds of $\mathcal{N} = 4$ SYM, realized as the low energy limit of the theory on N D3 branes at the tip of the cone \mathbb{R}^6/Γ . They found a one-to-one correspondence between double-trace couplings with negative discriminant and twisted tachyons in the tree-level spectrum of the type IIB background before the decoupling limit, namely $\mathbb{R}^{3,1} \times \mathbb{R}^6/\Gamma$. (Note that these flat-space tachyons are conceptually distinct from the tachyons in the curved $AdS_5 \times S^5/\Gamma$ background that have been the focus of this paper.)⁹ It

⁹The correspondence between twisted sector tachyons and field theory instabilities was first observed in [33] in the context of non-commutative field theory.

turns out that for *all* non-supersymmetric examples in this class, at least one double-trace coupling has negative discriminant, and conformal invariance is broken.

It will be interesting to investigate more general constructions to see if conformal examples exist, both as a question of principle and in view of phenomenological applications.¹⁰ One possibility, suggested by the correspondence found in [11], is to add discrete torsion in a way that removes the tree-level tachyons [36]. Another is to add appropriate orientifold planes. A promising candidate for a conformal orientifold theory is the $U(N)$ gauge theory with six scalars in the adjoint and four Dirac fermions in the antisymmetric representation of the gauge group [37].

Another important question, which is being investigated by [32], is to analyze the IR fate of non-supersymmetric orbifolds of $\mathcal{N} = 4$ SYM, by expanding their lagrangian around the local minimum of the effective potential. This is a well-posed field theory problem because the minimum can be trusted for small coupling. It would also be very interesting to extend the calculations of [10, 11] to two loops. At one-loop, there is no obvious distinction between freely acting and non-freely acting examples. This distinction may arise at two loops, with the freely acting cases beginning to show the behavior of figure 1.

Finally, it would be nice to find a more detailed AdS interpretation for the individual terms appearing in the double-trace beta function. For $\lambda = 0$, when only the term vf^2 is present, β_f can be reproduced by a simple bulk calculation [24], using the interpretation [24, 25] of the double-trace deformation as a mixed boundary condition for the bulk scalar. There should be a bulk interpretation for the other terms of β_f as well, in particular for the coefficient $a(\lambda)$ which drives the instability.

Acknowledgments

It is a pleasure to thank Igor Klebanov, Andrei Parnachev, Martin Rocek for very useful discussions. We thank Adi Armoni for useful correspondence on orientifold field theories and for pointing out the omission of the important reference [33] in the first version of this paper. The work of L.R. is supported in part by the National Science Foundation Grant No. PHY-0354776 and by the DOE Outstanding Junior Investigator Award. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

- [1] A. Parkes and P.C. West, *Finiteness in rigid supersymmetric theories*, *Phys. Lett. B* **138** (1984) 99 [SPIRES];
 P.C. West, *The Yukawa β -function in $N = 1$ rigid supersymmetric theories*, *Phys. Lett. B* **137** (1984) 371 [SPIRES];
 D.R.T. Jones and L. Mezincescu, *The chiral anomaly and a class of two loop finite supersymmetric gauge theories*, *Phys. Lett. B* **138** (1984) 293 [SPIRES];
 A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Finite $N = 1$ supersymmetric Grand Unified Theories*, *Nucl. Phys. B* **281** (1987) 72 [SPIRES];

¹⁰See e.g. [34, 35] for an approach to conformal phenomenology.

- D.I. Kazakov, *Finite $N = 1$ SUSY field theories and dimensional regularization*, *Phys. Lett. B* **179** (1986) 352 [SPIRES];
- D.R.T. Jones, *Coupling constant reparametrization and finite field theories*, *Nucl. Phys. B* **277** (1986) 153 [SPIRES];
- R. Oehme, *Reduction and reparametrization of Quantum Field Theories*, *Prog. Theor. Phys. Suppl.* **86** (1986) 215 [SPIRES];
- C. Lucchesi, O. Piguet and K. Sibold, *Necessary and sufficient conditions for all order vanishing β -functions in supersymmetric Yang-Mills theories*, *Phys. Lett. B* **201** (1988) 241 [SPIRES];
- X.-D. Jiang and X.-J. Zhou, *A criterion for existence of finite to all orders $N = 1$ SYM theories*, *Phys. Rev. D* **42** (1990) 2109 [SPIRES].
- [2] T. Banks and A. Zaks, *On the phase structure of vector-like gauge theories with massless fermions*, *Nucl. Phys. B* **196** (1982) 189 [SPIRES].
- [3] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].
- [4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109] [SPIRES].
- [5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [SPIRES].
- [6] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183] [SPIRES].
- [7] A.E. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four dimensions*, *Nucl. Phys. B* **533** (1998) 199 [hep-th/9803015] [SPIRES].
- [8] M. Bershadsky, Z. Kakushadze and C. Vafa, *String expansion as large- N expansion of gauge theories*, *Nucl. Phys. B* **523** (1998) 59 [hep-th/9803076] [SPIRES].
- [9] A. Adams and E. Silverstein, *Closed string tachyons, AdS/CFT and large- N QCD*, *Phys. Rev. D* **64** (2001) 086001 [hep-th/0103220] [SPIRES].
- [10] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative search for fixed lines in large- N gauge theories*, *JHEP* **08** (2005) 011 [hep-th/0505099] [SPIRES].
- [11] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative gauge theory and closed string tachyons*, *JHEP* **11** (2005) 038 [hep-th/0509132] [SPIRES].
- [12] M. Bershadsky and A. Johansen, *Large- N limit of orbifold field theories*, *Nucl. Phys. B* **536** (1998) 141 [hep-th/9803249] [SPIRES].
- [13] A.A. Tseytlin and K. Zarembo, *Effective potential in non-supersymmetric $SU(N) \times SU(N)$ gauge theory and interactions of type 0 D3-branes*, *Phys. Lett. B* **457** (1999) 77 [hep-th/9902095] [SPIRES].
- [14] C. Csáki, W. Skiba and J. Terning, *β -functions of orbifold theories and the hierarchy problem*, *Phys. Rev. D* **61** (2000) 025019 [hep-th/9906057] [SPIRES].
- [15] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett. B* **115** (1982) 197 [SPIRES].

- [16] O. Lunin and J.M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)] [[SPIRES](#)].
- [17] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [[hep-th/0503201](#)] [[SPIRES](#)].
- [18] S. Ananth, S. Kovacs and H. Shimada, *Proof of ultra-violet finiteness for a planar non-supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **783** (2007) 227 [[hep-th/0702020](#)] [[SPIRES](#)].
- [19] E. Pomoni and L. Rastelli, to appear.
- [20] G.T. Horowitz, J. Orgera and J. Polchinski, *Nonperturbative instability of $AdS_5 \times S^5/Z_k$* , *Phys. Rev. D* **77** (2008) 024004 [[arXiv:0709.4262](#)] [[SPIRES](#)].
- [21] K. Copley and R.B. Mann, *States of negative energy and $AdS_5 \times S_5/Z_k$* , *JHEP* **05** (2008) 069 [[arXiv:0803.3801](#)] [[SPIRES](#)].
- [22] O. Aharony, M. Berkooz and E. Silverstein, *Multiple-trace operators and non-local string theories*, *JHEP* **08** (2001) 006 [[hep-th/0105309](#)] [[SPIRES](#)].
- [23] O. Aharony, M. Berkooz and E. Silverstein, *Non-local string theories on $AdS_3 \times S^3$ and stable non-supersymmetric backgrounds*, *Phys. Rev. D* **65** (2002) 106007 [[hep-th/0112178](#)] [[SPIRES](#)].
- [24] E. Witten, *Multi-trace operators, boundary conditions and AdS/CFT correspondence*, [hep-th/0112258](#) [[SPIRES](#)].
- [25] M. Berkooz, A. Sever and A. Shomer, *Double-trace deformations, boundary conditions and spacetime singularities*, *JHEP* **05** (2002) 034 [[hep-th/0112264](#)] [[SPIRES](#)].
- [26] P. Minces, *Multi-trace operators and the generalized AdS/CFT prescription*, *Phys. Rev. D* **68** (2003) 024027 [[hep-th/0201172](#)] [[SPIRES](#)].
- [27] D.Z. Freedman, K. Johnson and J.I. Latorre, *Differential regularization and renormalization: a new method of calculation in Quantum Field Theory*, *Nucl. Phys. B* **371** (1992) 353 [[SPIRES](#)].
- [28] D.Z. Freedman, K. Johnson, R. Muñoz-Tapia and X. Vilasis-Cardona, *A cutoff procedure and counterterms for differential renormalization*, *Nucl. Phys. B* **395** (1993) 454 [[hep-th/9206028](#)] [[SPIRES](#)].
- [29] J.I. Latorre, C. Manuel and X. Vilasis-Cardona, *Systematic differential renormalization to all orders*, *Ann. Phys.* **231** (1994) 149 [[hep-th/9303044](#)] [[SPIRES](#)].
- [30] S.R. Coleman and E.J. Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, *Phys. Rev. D* **7** (1973) 1888 [[SPIRES](#)].
- [31] N. Beisert and R. Roiban, *The Bethe ansatz for $Z(S)$ orbifolds of $N = 4$ super Yang-Mills theory*, *JHEP* **11** (2005) 037 [[hep-th/0510209](#)] [[SPIRES](#)].
- [32] A. Dymarsky, S. Franco, I.R. Klebanov and R. Roiban, work in progress.
- [33] A. Armoni, E. Lopez and A.M. Uranga, *Closed strings tachyons and non-commutative instabilities*, *JHEP* **02** (2003) 020 [[hep-th/0301099](#)] [[SPIRES](#)].
- [34] P.H. Frampton and C. Vafa, *Conformal approach to particle phenomenology*, [hep-th/9903226](#) [[SPIRES](#)].

- [35] P.H. Frampton and T.W. Kephart, *Quiver gauge theory and conformality at the Large Hadron Collider*, *Phys. Rept.* **454** (2008) 203 [[arXiv:0706.4259](#)] [[SPIRES](#)].
- [36] Z. Kakushadze, *Tachyon-free non-supersymmetric strings on orbifolds*, *Int. J. Mod. Phys. A* **23** (2008) 4371 [[arXiv:0711.4108](#)] [[SPIRES](#)].
- [37] A. Armoni, *Non-perturbative planar equivalence and the absence of closed string tachyons*, *JHEP* **04** (2007) 046 [[hep-th/0703229](#)] [[SPIRES](#)].